On a theorem of L. Fejér concerning trigonometric interpolation

By PAUL SZÁSZ in Budapest

To my friend Professor L. Rédei on the occasion of his sixtieth birthday

It is well known that for any trigonometric polynomial

$$\varphi(\vartheta)=a_0+a_1\cos\vartheta+b_1\sin\vartheta+\cdots+a_n\cos n\vartheta+b_n\sin n\vartheta$$

of order $\leqq n$ the values

$$\varphi(\vartheta_0^*)=y_0,\ \varphi(\vartheta_1^*)=y_1,\ldots,\varphi(\vartheta_n^*)=y_n$$

of $\varphi(\vartheta)$ at the points

$$\vartheta_0^*=\tau,\quad \vartheta_1^*=\tau+\frac{2\pi}{n+1},\ldots,\ \vartheta_n^*=\tau+n\frac{2\pi}{n+1}$$

determine the value of the integral $\int_0^{2\pi}\varphi(\vartheta)\,d\vartheta$ uniquely, namely we have

$$\int_0^{2\pi}\varphi(\vartheta)\,d\vartheta=2\pi\frac{y_0+y_1+\cdots+y_n}{n+1}.$$

L. FEJÉR has stated without proof[1]) that this property for the point-system ϑ_k^* is characteristic. More precisely, the following statement holds:

Let

$$\vartheta_0<\vartheta_1<\cdots<\vartheta_n<\vartheta_0+2\pi,$$

and suppose that for any trigonometric polynomial $\varphi(\vartheta)$ of order $\leqq n$ with real coefficients, the conditions

$$\varphi(\vartheta_0)=0,\ \varphi(\vartheta_1)=0,\ldots,\varphi(\vartheta_n)=0$$

[1]) See L. FEJÉR, Über Interpolation, *Nachrichten der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-physikalische Klasse,* **1916**, 66—91, in particular p. 91.

imply $\int_0^{2\pi} \varphi(\vartheta)\,d\vartheta = 0$. *Then we have necessarily*

$$\vartheta_0 = \tau,\ \vartheta_1 = \tau + \frac{2\pi}{n+1}, \ldots, \vartheta_n = \tau + n\frac{2\pi}{n+1}$$

with some real number τ.

Let us be permitted to communicate here an easy proof for this theorem of FEJÉR.

Consider the polynomial

$$g(z) = (z-z_0)(z-z_1)\cdots(z-z_n) = z^{n+1} + a_1 z^n + \cdots + a_{\nu+1} z^{n-\nu} + \cdots + a_{n+1},$$

where

$$z_k = \cos\vartheta_k + i\sin\vartheta_k \qquad (k = 0, 1, \ldots, n),$$

and put

$$g_\nu(z) = z^{\nu-n} g(z) \qquad (\nu = 0, 1, \ldots, n-1).$$

Then $\chi_\nu(\vartheta) = g_\nu(\cos\vartheta + i\sin\vartheta)$ is a trigonometric polynomial of order $\leqq n$ with the absolute member $a_{\nu+1}$. Let $\varphi_\nu(\vartheta)$ and $\psi_\nu(\vartheta)$ be the real and imaginary parts of $\chi_\nu(\vartheta)$, these are trigonometric polynomials of order n at most with real coefficients. Since $\chi_\nu(\vartheta_k) = z_k^{\nu-n} g(z_k) = 0$, thus

$$\varphi_\nu(\vartheta_k) = 0,\ \psi_\nu(\vartheta_k) = 0 \qquad (k = 0, 1, \ldots, n),$$

so we have by assumption

$$\int_0^{2\pi} \varphi_\nu(\vartheta)\,d\vartheta = 0,\quad \int_0^{2\pi} \psi_\nu(\vartheta)\,d\vartheta = 0,$$

i. e. the polynomials $\varphi_\nu(\vartheta)$, $\psi_\nu(\vartheta)$ have their absolute members equal to zero. Thus we have

$$a_{\nu+1} = 0 \qquad (\nu = 0, 1, \ldots, n-1),$$

i. e.

$$g(z) \equiv z^{n+1} + a_{n+1}.$$

Consequently the roots of $g(z)$ form a regular $(n+1)$-angle inscribed in the unit circle. This proves the theorem.

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